GLOBAL BEHAVIOR OF SOLUTIONS OF NONLINEAR ODES: FIRST ORDER EQUATIONS

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ABSTRACT. We determine the behavior of the general solution, small or large, of nonlinear first order ODEs in a neighborhood of an irregular singular point chosen to be infinity. We show that the solutions can be controlled in a ramified neighborhood of infinity using a finite set of asymptotic constants of motion; the asymptotic formulas can be calculated to any order by quadratures. These constants of motion enable us to obtain qualitative and accurate quantitative information on the solutions in a neighborhood of infinity, as well as to determine the position of their singularities. We discuss how the method extends to higher order equations. There are some conceptual similarities with a KAM approach, and we discuss this briefly.

1. Introduction

The point at infinity is most often an *irregular singular point* for equations arising in applications.[1] Within this class of equations, there are essentially two types for which a global description of solutions exists: linear systems and integrable ones. However, in a stricter sense, even for some linear problems global questions such as explicit values of connection coefficients are still open. The behavior of the general solutions of *linear* ODEs has been thoroughly analyzed starting in the late 19th century, see [24] and [38] and references therein. After the pioneering work of Écalle, Ramis, Sibuya and others the description of their solutions in $\mathbb{C}$ is by now quite well understood [23, 22, 5, 6, 10, 33, 34, 16].

Integrable systems provide another important class of systems allowing for global description of solutions. The ensemble of integrable systems is a zero measure set in the parameter space of general equations: a generic small perturbation of an integrable system destroys integrability. Nonetheless, integrable equations occur remarkably often in many areas of mathematics, such as orthogonal polynomials, the analysis of the Riemann-zeta function, random matrix theory, self-similar solutions of integrable PDEs and combinatorics, cf. [8],[18]–[20], [1, 11, 12, 14, 20], [21]–[40]. However, even in integrable systems, achieving global control of solutions in a *practical way* is a challenging task, and it is one of the important aims of the emerging Painlevé project [9].

In *nonintegrable* systems, particularly near irregular singularities, our understanding is much more limited. Small solutions are given by generalized Borel summable *transseries*; this was discovered by Écalle in the 1980s and proved rigorously

[1] A singular point of an equation is irregular if, for *small* solutions, the linearization is not of Frobenius type. By a small solution we mean one that tends to zero in some direction after simple changes of coordinates.

in many contexts subsequently. Transseries are essentially formal multiseries in powers of $1/x^{k_i}\ e^{-\lambda_j x}$, and possibly $x^{-1}\log x$; see again [23, 22, 5, 6, 10, 33, 34, 16] and [15]. Here x is the independent variable and λ_j are eigenvalues of the linearization with the property $\mathrm{Re}\,(\lambda_j x) > 0$. In general, *only* small solutions are well understood. However, for generic nonlinear systems of higher order, small solutions form lower dimensional manifolds in the space of all solutions, see, e.g., [16]. The present understanding of general nonlinear equations is thus quite limited.

We introduce a new line of approach, combining ideas from generalized Borel summability and KAM theory (see, e.g. [3]) for the analysis near infinity, chosen to be an irregular singular point, of solutions of relatively general differential equations with meromorphic coefficients. Applying the method does not require knowledge of Borel summability, transseries or KAM theory.

For small solutions, in [17] it was shown that in a region adjacent to the sector where the solution, y, is small, $y(x)$ is almost periodic. In this sense y becomes an approximately cyclic variable. In the x-complex plane, the singular points of y are arranged in quasi-periodic arrays as well. The analysis in [17] covers an angularly small region beyond the sector where y is small. Looking directly at the asymptotics of y beyond this region would require a multiscale approach: y has a periodic behavior–the fast scale, with $O(1/x)$ changes in the quasi-period. Multiscale analysis is usually a quite involved procedure (see, e.g., [7]).

It is natural to make a hodograph transformation in which the dependent and independent variables are switched. As mentioned above, in the "nontrivial" regions, the dependent variable is an almost cyclic one. The setting becomes somewhat similar to a KAM one: there is an underlying completely integrable system, and one looks for persistence of invariant tori. Adiabatic invariants are simply the conserved quantities associated with these tori. Evidently there are many differences between the ODE setting and the KAM one, for instance the fact that the small parameter is "internal", $1/x$.

In this work we restrict the analysis to first order equations, mainly to ensure a transparent and concrete analysis. In theory however, the method generalizes to equations of any order, and we touch on these issues at the end of the paper.

We look at equations which, after normalization, are of the form $dy/dx = F(z, y)$, $z = 1/x$, with g bi-analytic at $\mathbf{0}$ and $F_y(\mathbf{0}) = 1$.

We show that in any sector on Riemann surfaces towards infinity, the *general* solution is represented by transseries and/ or, in an implicit form, by some constant of motion. In fact, on large circles around $x = 0$, the solution cycles among transseries representations and ones in which constants of motion describe it accurately. The regions where these behaviors occur overlap slightly to allow for asymptotic matching (cf. Corollary 8). The connection between the large x behavior and the initial condition is relatively easy to obtain.

Let $\beta = F_{zy}(\mathbf{0})$. The constants of motion have asymptotic expansions of the form

$$C(x, y) \sim x - \beta\log x + F_0(y) + x^{-1}F_1(y) + \cdots + x^{-j}F_j(y) + \cdots, \quad x \to \infty. \tag{1}$$

Clearly, under the assumptions above, the solution y can be obtained asymptotically from (1) and the implicit function theorem. The requirement that C is to leading order of the form $f_1(x) + f_2(y)$, determines C up to trivial transformations, see Theorem 5 and Note 2.

The functions F_j are shown in the proof of Theorem 5 to solve first order autonomous ODEs, and thus they can always be calculated by quadratures.

To illustrate this, we use a nonintegrable Abel equation,

$$u' = u^3 - t. \tag{2}$$

We note that there is no consensus on how nonintegrabilty should be defined; for (2), it is the case that the equation passes no criterion of integrability, including the poly-Painlevé test, and that there are no solutions known, explicit or coming from, say, some associated Riemann-Hilbert reformulation.

The Abel equation has the normal form (see §4, where further details about this example are given)

$$y' + 3y^3 - \frac{1}{9} + \frac{1}{5x}y = 0. \tag{3}$$

Regions of smallness are those for which y approaches a root of $3y^3 - 1/9$; in these regions, y is given by a transseries [17]. Otherwise, y has an implicit representation of the form

$$y = \frac{1}{3}\exp\Bigg(- C - x + \frac{1}{5}\log x + \left(\sqrt{3} - \frac{2\sqrt{3}}{5x}\right)\arctan\left(\frac{6y+1}{\sqrt{3}}\right) \tag{4}$$
$$- \log(3y-1) + \frac{1}{2}\log(9y^2+3y+1)) + \frac{1}{x}\left(\frac{27y^2}{5(1-27y^3)} + \frac{1}{25} + O(1/x)\right)\Bigg) + \frac{1}{3},$$

obtained by inverting an appropriate constant of motion C (see (39)); for the values of β, F_0, F_1 see §4.1.

While in a numerical approach to calculating solutions the precision deteriorates as x becomes large, the accuracy of (1) instead, *increases.* In examples, even when (1) is truncated to two orders, (1) is strikingly close to the actual solution even for relatively small values of the independent variable, see e.g. Figure 4.

The procedure allows for a convenient way to link initial conditions to global asymptotic behavior, see e.g. (41).

1.1. **Solvability versus integrability.** First order equations for which the associated second order autonomous system is Hamiltonian are in particular *integrable.* Indeed, by their definition, there is a globally defined smooth H with the property that $\dot{x}\frac{\partial H}{\partial x} + \dot{y}\frac{\partial H}{\partial y} = 0$, that is $H(x(t), y(t)) = const$, providing a closed form implicit, global representation of y. While the differential equation provides "infinitesimal" information, H –effectively an integral– provides a global one.

Conversely, clearly, if there exists an implicit solution of the equation or indeed a smooth enough conserved quantity, the equation comes from a Hamiltonian system.

What we provide is a finite set of matching conserved quantities, analogous to an atlas of overlapping maps projecting the differential field onto the trivial one, $H' = 0$. They give, in a sense, a *foliation of the phase space* allowing for global control of solutions. With obvious adaptations, this picture extends to higher order systems. In integrable systems there is just one single-valued map and the field is globally rectifiable. In general, the conserved quantities may be branched and not globally defined.

1.2. **Normalization and definitions.** Many equations of the form $y' = F(y, 1/x)$ with F analytic for small y and small $1/x$ can be brought to the normal form $y' = P_0(y) + Q(y, 1/x)$ by systematic changes of variables, see *e.g.* [16], [15].

The assumptions are that $Q(y, z)$ is entire in y and analytic in z for small z, and $O(y^2, yz^2, z)$ for small y and z and that P_0 is a polynomial. We assume that the roots of P_0 are *simple*. It will be seen from the analysis that a more general P_0 can be accommodated. We thus write the equation as

$$y' = \sum_{k=0}^{\infty} \frac{P_k(y)}{x^k} = Q_1(y, 1/x) = P_0(y) + Q(y, 1/x). \tag{5}$$

Definition 1. • *A formal constant of motion of* (5) *for $x \to \infty$ in an unbounded domain $\mathcal{D} \subset \mathbb{C}^2$ or on a Riemann surface covering it, and in which to leading order in $1/x$ the variables x and y are separated additively is a formal series*

$$\tilde{C}(y, x) = A(x) + F_0(y) + \frac{F_1(y)}{x} + \cdots + \frac{F_j(y)}{x^j} + \cdots \tag{6}$$

such that we have

$$\frac{d}{dx}\tilde{C}(y(x), x) = O(x^{-\infty})$$

in the sense that, for any j, F_j and H_j defined by

$$\frac{H_{j+1}(x, y)}{x^{j+1}} := A'(x) + D_x \left(F_0(y) + \frac{F_1(y)}{x} + \cdots + \frac{F_j(y)}{x^j} \right) \tag{7}$$

are uniformly bounded in $\mathcal{D}$; here D_x is the derivative along the field,

$$D_x F(x, y) = \nabla F \cdot (1, Q_1) = F_x(x, y) + F_y(x, y) Q_1(y, 1/x).$$

See also (10) *below.*

• *An actual constant of motion associated to $\tilde{C}$ in $\mathcal{D} \subset \mathbb{C}^2$ is a function C so that $C(y, x) \sim \tilde{C}(y, x)$ as $x \to \infty$ and $\frac{d}{dx} C(y(x), x) = 0$ for all solutions in $\mathcal{D}$.*

Note 2. *It will be seen that there is rigidity in the form of the constant of motion: if the variables in $\tilde{C}$ are, to leading order, separated additively as in* (6)*, then, up to trivial transformations, we must have*

$$A(x) = -x + a \log x \tag{8}$$

where a is the same as the one in the transseries expansion of the solution, see Proposition 6.

1.3. **Finding the terms in the expansion of $\tilde{C}$.** Using (8) and truncating (6) at an arbitrary $n > 2$, let

$$C_n(y, x) =: -x + a \log x + F_0(y) + \sum_{k=1}^{n} \frac{F_k(y)}{x^k}. \tag{9}$$

We can check that $D_x C_n$ satisfies

$$\begin{aligned} D_x C_n = {}& -1 + P_0 F_0' + \frac{a + P_1 F_0' + P_0 F_1'}{x} \\ &+ \sum_{k=2}^{n} \frac{(1-k)F_{k-1} + \sum_{j=0}^{k} P_{k-j} F_j'}{x^k} + \frac{-nF_n + \sum_{j=0}^{n} \sum_{k=0}^{\infty} P_{n+k+1-j} F_j' x^{-k}}{x^{n+1}} \end{aligned} \tag{10}$$

(cf. (5)) where the numerator of the last term is H_n by definition. In order for $\tilde{C}$ to be a formal constant of motion, the coefficients of $x^{-j}, j = 0, 1, 2, \ldots$ must vanish, giving

$$F_0'(y) = \frac{1}{P_0(y)} \tag{11}$$

$$F_1'(y) = -\frac{a + P_1(y)F_0'(y)}{P_0(y)} \tag{12}$$

$$F_k'(y) = \frac{(k-1)F_{k-1}(y) - \sum_{j=0}^{k-1} P_{k-j}(y)F_j'(y)}{P_0(y)} \qquad (2 \le k \le n). \tag{13}$$

It follows in particular that $F_0' \neq 0$ and F_0 is bounded in $\mathcal{D}$. In solving the differential system, the constants of integration are chosen so that F_k are indeed uniformly bounded in y, see (25).

1.4. **Solving for $y(x)$.** The expression C_n is an approximate constant of motion ; we thus can find an approximate solution y_n by fixing $C_n = K$. We then write

$$G(y;K) := F_0(y) - K - x + a\log x + \sum_{k=1}^{n} \frac{F_k(y)}{x^k} = 0 \tag{14}$$

and we note that in the domain relevant to us ($\mathcal{S}_1$, see Theorem 4 below) the analytic implicit function theorem applies since

$$\frac{\partial G}{\partial y} = \frac{1}{P_0(y)} + \frac{1}{x}E_1(y,x) \tag{15}$$

where P_0 is away from 0 in our domain, and for some E_1 which is bounded in $\mathcal{D}$ by (9) since Y is bounded. Writing $y = y_n$ in (14) and using the analytic implicit function theorem, treating $1/x$ as a small parameter, we get

$$y_n = G_0(x;K) + \frac{G_1(x;K)}{x} + \cdots + \frac{G_n(x;K)}{x^n} + \frac{\tilde{H}_n(x;K)}{x^{n+1}} \tag{16}$$

where $\tilde{H}_n$ and the G_j's are bounded. In the same way it is checked that y_n is solution of (5) up to corrections $R_n(x;K)/x^{n+1}$, that is, $y_n' - Q_1(y_n, 1/x) = -R_n(x;K)/x^{n+1}$ where R_n is bounded.

Let $p_1, \ldots, p_m$ be the distinct roots of P_0.

Let $\mathcal{R}_y$ be the universal cover of $Y = \mathbb{C}\backslash\{p_1, ..., p_m\}$. Let $\pi : X \to Y$ be the covering map.

Definition 3. • An **elementary y-path** of type

$$\alpha = (\alpha_1, ..., \alpha_m, \alpha_{m+1}, \ldots, \alpha_{mk}) \in \mathbb{Z}^{mk}, k \in \mathbb{N}$$

is a piecewise smooth curve γ in $\mathcal{R}_y$ whose image under π turns α_1 times around p_1, then α_2 times around p_2, and so on, α_m times around p_m, then again α_{m+1} times around p_1 , etc. Note that α is in fact an element of the fundamental group.

• A **y-path of type $\boldsymbol{\alpha}$** is a smooth curve γ obtained as an arbitrary forward concatenation of elementary y-paths of type α. More precisely, a y-path of type α is a map $\gamma : [0, \infty) \to \mathcal{R}_y$ so that, for any $N \in \mathbb{Z}^+$, $\gamma|_{[N,N+1]}$ is an elementary y-path of type α. We will naturally denote by $\gamma|_{[0,a]}$ subarcs of γ. We see that y-paths are compositions of *closed loops* in the *complex y domain.*

• $\mathcal{S}_r$ is **a regular domain of type $\boldsymbol{\alpha}$** or an R-domain of type α, if it is an unbounded open subset of $\mathcal{R}_y$ that contains only images of y-paths of type α. Thus the image of any unbounded y-path of type $\alpha' \neq \alpha$ is not a subset of $\mathcal{S}_r$.

Note. In our results we only need y-paths with the additional property that $x(y) \to \infty$ along the path.

To take a trivial illustration, in the equation $y' = y$ an example of a y-path along which $x \to \infty$ is $t \mapsto \exp(it), t \geq 0$.

2. Main results

2.1. **Existence of formal constants of motion.** Under the assumptions at the beginning of §1.2 we have

Theorem 4. *Let $\mathcal{S}_y$ be an R-domain of type α, and*

$$\mathcal{S}_1 = \{y \in \mathcal{S}_y : |\pi(y)| < M_0 \text{ and } |\pi(y) - p_k| > \epsilon \text{ for all } k\}$$

where $M_0 > 0$ is an arbitrary constant. Let $\mathcal{C}$ be the union of m circular paths surrounding $\alpha_k \in \mathbb{Z}$ times the root p_k, $k = 1, \ldots, m$, chosen so that

$$\int_{\mathcal{C}} \frac{1}{P_0(y)} dy \neq 0. \tag{17}$$

Then, if R_0 is large enough, there exists a formal constant of motion in

$$\mathcal{D}_1 = \{(x, y) : |x| > R_0, y \in \mathcal{S}_1\}$$

of the form (6)*. The terms F_k in the expansion of $\tilde{C}$ in* (6) *can be calculated by quadratures.*

Actual constants of motion are obtained in Theorem 5.

Consider now a set $\mathcal{S}$ of curves γ, $|\gamma(t)| \to \infty$ as $t \to \infty$, with the property that for all $t_1 < t_2$ and all n (which is in fact equivalent to for $n = 0, 1$)

$$\left| \mathrm{Re} \int_{\gamma(t_1)}^{\gamma(t_2)} \frac{\partial}{\partial y} Q_1(y, \gamma(t))|_{y = y_n(\gamma(t))} \gamma'(t) dt \right| \leqslant b \log(|\gamma(t_2)/\gamma(t_1)| + 1), \tag{18}$$

where $b > 0$ is a constant, and such that there is an M so that for all n we have $|y_n| < M$ along γ. Here M can be chosen large if x is large. Note that $\mathcal{S}$ contains the curves $\gamma(t)$ so that $y_n(\gamma(t))$ is an α-path. Indeed, by (16) , in this case, the integrand in (18) is of the form $\frac{P_0'(y)}{P_0(y)} dy + O(1) \frac{d\gamma(t)}{\gamma(t)}$ and hence the integral equals $2\pi i N + O(\log(|N| + 1))$ for large N where N is the number of loops.

Theorem 5. *Assume $\tilde{C}$ in* (6) *is a formal constant of motion in a region $\mathcal{D} = \mathcal{S} \cap \mathcal{D}_1$. Then there exists an actual constant of motion $C = C(x, y)$ defined in the same region, so that $C \sim \tilde{C}$ as $x \to \infty$.*

2.2. **Regions where $P_0(u)$ is small.** Assume x_0 is large and $|P_0(y(x_0))| < \epsilon$ is sufficiently small. This means that for some root r_k of P_0 we have $|y(x_0) - r_k| < \epsilon_1$ where ϵ_1 is also small. Without loss of generality we can assume that $r_k = 0$ and $x \in \mathbb{R}^+$ since the change of variables $y_1 = y - r_k$, $x = x_1 e^{i\phi}$ does not change the form of the equation. Assume also that after normalization the stability condition $\mathrm{Re}\, P_0'(0) < 0$ holds. Again without loss of generality, by taking $y_2 = \alpha y_1$ we can arrange that $P_0'(0) = -1$. The new function Q in (5) will have the form

$y^2Q_1(y,1/x)+x^{-2}Q_2(y,1/x)$ where Q_1 and Q_2 are analytic for small y and $1/x$. As a result, the normalized equation assumes the form

$$y' = -y + f_0(x) + \frac{ay}{x} + y^2Q_1(y,1/x) + x^{-2}Q_2(y,1/x). \tag{19}$$

We also arrange that $f_0 = O(x^{-M})$ as $x \to \infty$, for suitably large M; this is possible through a change of variables of the form $y_2 = y_3 + \sum_{k=1}^{M} c_k x^{-k}$, where the c_k's are the coefficients of the formal power series solution for small y.

Proposition 6. *[see* [16] *Theorem 3] Any solution of* (19) *that is* $o(1)$ *as* $x \to \infty$ *along some ray in the right half plane can be written as a Borel summed transseries, that is*

$$y(x) = \sum_{k=0}^{\infty} C^k x^{ka+1} e^{-kx} y_k \tag{20}$$

where y_k *are generalized Borel sums of their asymptotic series, and the decomposition is unique. There exist bounds, uniform in* n *and* x*, of the form* $|y_n(x)| < A^k$*, and thereby the sum converges uniformly in a region* $\mathcal{R}$ *that contains any sector* $\mathcal{S}_c := \{x : |\arg x| < c < \pi/2\}$*. Note that Theorem 3 in* [16] *applies to general* n*-th order ODEs.*

Proposition 7. *(i) If, after the normalization above,* $y(x_0)$ *is small (estimates can be obtained from the proof), then* y *is given by* (20).

(ii) $C(y(x),x)$*, obtained by inversion of (20) for large* x *in the right half plane and small* y*, is a constant of motion defined for all solutions for which* $y(x_0)$ *is small (cf. (i)).*

Proof. (i) We write the differential equation in the equivalent integral form

$$\begin{aligned} y = {} & F_0(x) + y_0 e^{-(x-x_0)}(x/x_0)^a \\ & + e^{-x}x^a \int_{x_0}^{x} e^s s^{-a}\left[y^2(s)Q_1(y(s),1/s) + s^{-2}Q_2(y(s),1/s)\right] ds, \end{aligned} \tag{21}$$

where $F_0(x) = O(x^{-M})$ (M can be chosen arbitrarily large in the normalization process, [16]) and $F_0(x_0) = 0$. It is straightforward to show that for (21) is contractive in the norm $\|y\| = \sup_{x\in\mathcal{S}_c} |x^{M-1}y(x)|$ (see the beginning of this section) and thus it has a unique solution in this space. Hence, by uniqueness, the solution of the ODE with $y(x_0) = y_0$, has the property $y(x) \to 0$ as $x \to \infty$. The rest of (i) now follows from [16].

(ii) We see from Proposition 6 that $y(x;C)$ is analytic in a domain of the form $\mathcal{S}_c \times \mathbb{D}_\rho$ (As usual, $\mathbb{D}_\rho$ denotes the disk of radius ρ.) We look at the rhs of (20) as a function $H(x,C)$. It follows from [16] that $y_1(x) = x^{-1}(1 + o(1/x))$. By uniform convergence, we clearly have

$$\frac{\partial H}{\partial C} = \sum_{k=0}^{\infty} kC^{k-1}x^{ka+1}e^{-kx}y_k = e^{-x}x^a(1+o(1)) \neq 0. \tag{22}$$

The rest follows from the implicit function theorem. $\square$

As a result of Theorem 5 and Proposition 7 we have the following:

Corollary 8. *If* G_0 *in* (16) *approaches a root of* P_0 *and* x *is large enough, then* y *enters a transseries region, where the new constant is given, after normalization, by Proposition 7 (ii); thus the constants of motion in different regions match.*

3. Proofs and further results

3.1. **Proof of Theorem 4.** Let $(x_0, y_0) \in \mathcal{D}_1$. Recalling (10), we see that (11) has the solution

$$F_0(y) = \int_{y_0}^{y} \frac{1}{P_0(s)} ds + c_0$$

(we take $c_0 = 0$ since it can be absorbed into the constant of motion). Eq. (12) gives

$$F_1(y) = f_1(y) + c_1 := -\int_{y_0}^{y} \frac{a + \frac{P_1(s)}{P_0(s)}}{P_0(s)} ds + c_1, \tag{23}$$

where to ensure boundedness of $F_1(y)$ as the number of loops $\to \infty$, we let

$$a = -\frac{\int_{\mathcal{C}} \frac{P_1(y)}{P_0(y)^2} dy}{\int_{\mathcal{C}} \frac{1}{P_0(y)} dy}$$

and c_1 is determined to ensure boundedness of F_2 (cf. (25)). Inductively we have (24)

$$F_{k+1}(y) = \int_{y_0}^{y} \frac{k(f_k(s) + c_k) - \sum_{j=0}^{k} P_{k+1-j}(s) F_j'(s)}{P_0(s)} ds + c_{k+1} =: f_{k+1}(y) + c_{k+1}$$

for $2 \le k+1 \le n$, and, to ensure boundedness of $F_{k+1}(y)$ as the number of loops $\to \infty$ we need to choose

$$c_k = \frac{\int_{\mathcal{C}} \dfrac{-k f_k + \sum_{j=0}^{k} P_{k+1-j}(y) f_j'(y)}{P_0(y)} dy}{k \int_{\mathcal{C}} \dfrac{1}{P_0(y)} dy} \tag{25}$$

for $1 \le k \le n-1$.

It is clear by induction that every singularity of $F_k(y)$ is a root of P_0. To complete the proof we need to show that the F_k's are bounded in $\mathcal{D}_1$:

Lemma 9. *Assume $y \in \mathcal{S}_1$. For $\deg(P_0) \ge 1$ and $1 \le k \le n$ we have*

$$|F_k'(y)| \lesssim k! \quad \text{and} \quad |F_k(y)| \lesssim k!(|y| + 1)$$

where, as usual, $\lesssim$ means $\le$ up to an irrelevant multiplicative constant.

Proof. We prove the lemma by induction on k. Note that in (23) and (24) the integration paths can be decomposed into finitely many circular loops $\mathcal{C}$ and a ray, slightly deformed around possible singularities, which implies

$$|F_1(y)| \lesssim \log|y| + 1 \lesssim |y| + 1$$

and

$$|F_k(y)| \lesssim \left| \int_{y_0}^{y} |F_k'(s)| ds \right|$$

where the integration path is a straight line (possibly bent as above).

We see from (13) that

$$|F_k'(y)| \lesssim \frac{(k-1)|F_{k-1}(y)|}{|P_0(y)|} + \sum_{j=0}^{k-1} |F_j'(y)| \lesssim \frac{(k-1)|F_{k-1}(y)|}{|y| + 1} + \sum_{j=0}^{k-1} |F_j'(y)|.$$

The conclusion then follows by induction. Note that the the last term of (10) satisfies

$$\left|-nF_n+\sum_{j=0}^{n}\sum_{k=0}^{\infty}P_{n+k+1-j}F_j'x^{-k}\right|\lesssim (n+1)!(|y|+1)|P_0(y)|$$

□

3.2. **Proof of Theorem 5.** Let $y(x;K)=y_n(x;K)+\delta(x;K)$, where y_n is given in (16). We seek δ so that y is an exact solution of (5) in $\mathcal{D}$.

Let $\phi(y,\delta)$ be the polynomial satisfying $Q_1(y+\delta,x)-Q_1(y,x)=Q_{1,y}(y,x)\delta+\delta^2\phi(y,\delta,x)$ where $Q_{1,y}(y,x):=\frac{\partial Q_1(y,x)}{\partial y}$. We obtain

$$\delta'-\frac{b\delta}{x}-\frac{\partial Q_1(y,x)}{\partial y}\delta=\frac{R(x;K)}{x^{n+1}}-\frac{b\delta}{x}+\phi(y_n,\delta,x)\delta^2=:E(x;\delta(x);K), \tag{26}$$

where $R=:R_n$ is defined after (16); both R and ϕ are, by assumption, bounded. In integral form, (26) reads

$$\delta(x)=\int_{\infty}^{x}\frac{x^b}{s^b}e^{\int_s^x Q_{1,y}(y_n(t),t)dt}E(s;\delta(s);K)ds \tag{27}$$

where the integrals are taken along curves in $\mathcal{D}$. Using (18) we see that (27) is contractive in the norm $\|\delta\|=\sup\limits_{|x|\geqslant|x_1|;x\in\mathcal{D}}|x|^n|\delta(x)|$ in an arbitrarily large ball, if $|x_1|$ is large enough and $n>b2^{b+1}$.

Thus (27) has a unique solution and, of course, $\delta(x)$ is the limit of the Picard like iteration

$$\begin{aligned}\delta_0&=\int_{\infty}^{x}\frac{x^b}{s^b}e^{\int_s^x Q_{1,y}(y_n(t),t)dt}\frac{R(s;K)}{s^{n+1}}ds\\ \delta_1&=\int_{\infty}^{x}\frac{x^b}{s^b}e^{\int_s^x Q_{1,y}(y_n(t),t)dt}E(s;\delta_0(s);K)ds\\ &\qquad\qquad\qquad\qquad\qquad\qquad\textit{etc.}\end{aligned} \tag{28}$$

By (27) δ is a smooth function depending on (x,K) only, and $\delta=O(x^{-n})$. Smoothness is shown as usual by bootstrapping the integral representation (27).

Now we have, by (15), $\partial_K y_n(x;K)=P_0(y_n)(1+o(1))$. We can easily check that $\partial_K\delta(x,K)=O(x^{-n})$. This is done using essentially the same arguments employed to check contractivity of the integral equation for δ in the equation in variations for δ_K, derived by differentiating (26) with respect to K. We use the implicit function theorem to solve for K, giving $K=K(x,y)$, a smooth function of (x,y). It has the following properties: $K(x,y(x))$ is by construction constant along admissible trajectories and by straightforward verification, i.e. comparing K with $\tilde{C}$, we see that it is asymptotic to $\tilde{C}$ up to $O(x^{-n})$. It is known that if a function differs from the nth truncate of its series by $O(x^{-n})$ for large n, then in fact the difference is $o(x^{-n})$ (cf. [15] Proposition 1.13 (iii)).

3.3. **Position of singularities of the solution.** It is convenient to introduce constants of motion specific to singular regions; they provide a practical way to determine the position of singularities, to all orders.

Definition 10. *We define a* **simple singular solution path** $\gamma(s) : [0,1) \to \mathcal{R}_y$ *to be a piecewise smooth curve whose projection* $\pi(\gamma[0,1)) \in \mathbb{C}$ *is unbounded but turns around every* p_k *only finitely many times.*

A **simple singular solution domain** $\mathcal{S}_s$ *is the homotopy class of any simple singular solution path, in the sense that any two unbounded paths in* $\mathcal{S}_s$ *can be continuously deformed into each other without passing through any* p_k.

Proposition 11. *Let* $m_0 = \deg(P_0) \ge 2$, $\mathcal{S}_s$ *be a simple singular solution domain, and* $\mathcal{D}_2 = \{(x,y) : |x| > R, y \in \mathcal{S}_s, \text{ and } |y - p_k| > \epsilon \text{ for all } k\}$. *Assume that*

$$\frac{|P_k(y)|}{|P_0(y)|} \lesssim |y|^{-q}$$

for large y, *for some* $q \ge 0$ *and all* $k \ge 1$. *Note that this needs only be true in* $\mathcal{S}_s$, *which could be an angular region.*

Then there exists in $\mathcal{D}_2$ *a formal constant of motion of the form*

$$\tilde{C}(y,x) = x + F_0(y) + \frac{F_1(y)}{x} + \cdots + \frac{F_j(y)}{x^j} + \cdots , \tag{29}$$

where $F_k(y)$ *are single valued as* $y \to \infty$. *Furthermore, any simple singular solution path passing through some arbitrary* (x_0, y_0) *tends to a singularity, whose position* x_{sing} *satisfies*

$$x_{sing} = C_n(y_0, x_0) + O\left(\frac{1}{x_0^{n+1}}\right) \tag{30}$$

for all $n \in \mathbb{N}$, *where* C_n *is* $\tilde{C}$ *truncated to* x^{-n}.

Moreover, if there are only finitely many nonzero P_k, *then there exists in* $\mathcal{D}_2$ *a true constant of motion of the form (29), i.e. the sum is convergent for large* $|x|$.

Proof. The proof is similar to that of Theorem 4.

In order for $\tilde{C}$ to be a formal constant of motion, we must have

$$F_0'(y) = -\frac{1}{P_0(y)} \tag{31}$$

$$F_k'(y) = \frac{(k-1)F_{k-1}(y) - \sum_{j=0}^{k-1} P_{k-j}(y) F_j'(y)}{P_0(y)} \quad (1 \le k \le n). \tag{32}$$

We solve successively for the F_k and obtain

$$F_0(y) = \int_\infty^y \frac{1}{P_0(s)} ds$$

where the integration path lies in $\mathcal{S}_s$. Clearly F_0 is bounded and single valued as $y \to \infty$.

Inductively we have

$$F_k(y) = \int_\infty^y \frac{(k-1)F_{k-1}(s) - \sum_{j=0}^{k-1} P_{k-j}(s) F_j'(s)}{P_0(s)} ds \tag{33}$$

for $1 \le k \le n$.

To prove the rest of the proposition, we need the following lemma:

Lemma 12. *Assume that* $y \in \mathcal{S}_s$. *For* $1 \le k \le n$ *we have*

$$|F_k'(y)| \lesssim \frac{k!}{|y|^{m_0+q}}$$

$$|F_k(y)| \lesssim \frac{k!}{|y|^{m_0+q-1}}$$

as $y \to \infty$.

Furthermore, if $P_k = 0$ *for* $k > k_0 > 0$*, then*

$$|F_k'(y)| \lesssim \frac{c^k}{|y|^{m_0+q}}$$

$$|F_k(y)| \lesssim \frac{c^k}{|y|^{m_0+q-1}}.$$

Proof. The estimates are obtained by induction on k. Note that (32) implies

$$|F_k'(y)| \lesssim \frac{(k-1)|F_{k-1}(y)|}{|y|^{m_0}} + |y|^{-q} \sum_{j=0}^{k-1} |F_j'(y)|$$

provided that the assumptions of the lemma hold for $1 \le j \le k-1$.

If $P_k = 0$ for $k > k_0 > 0$, we again show the lemma by induction. Assume that for $0 < l \le k$ we have

$$|F_l'(y)| \le (c_0 l_0)^l \sum_{j=1}^{l+1} \binom{l}{j-1} |y|^{-1+j(1-m_0)-q}$$

(this is obviously true for $l = 1$).

This implies

$$|F_k(y)| \le (c_0 k_0)^k \sum_{j=1}^{k+1} \binom{k}{j-1} \frac{|y|^{j(1-m_0)-q}}{j(m_0-1)+q}$$

Thus it follows from (32) that

$$F_{k+1}'(y) = \frac{kF_k(y) - \sum_{j=\max\{k-k_0+1,0\}}^{k} P_{k+1-j}(y) F_j'(y)}{P_0(y)}$$

where, by the induction assumption, the first term satisfies the estimate

$$\begin{aligned}
\left|\frac{kF_k(y)}{P_0(y)}\right| &\le c_1 k \left|\frac{F_k(y)}{y^{m_0}}\right| \\
&\le c_0^k k_0^{k+1} c_1 \sum_{j=2}^{k+2} \frac{(k+1)\binom{k}{j-2}}{(j-1)(m_0-1)} |y|^{-1+j(1-m_0)-q} \\
&\le c_0^k k_0^{k+1} c_1 \sum_{j=2}^{k+2} \binom{k+1}{j-1} |y|^{-1+j(1-m_0)-q}
\end{aligned} \tag{34}$$

where $c_0 > 1 + c_1$. Note that the last inequality follows from

$$(k+1)\binom{k}{j-2} = \binom{k+1}{j-1}(j-1).$$

The second term is easy to estimate, since it is clearly bounded by

$$k_0(c_0 k_0)^k \sum_{j=1}^{k+1} \binom{k}{j-1} |y|^{-1+j(1-m_0)-q}.$$

Since c_1 is fixed, we can assume that $c_0 > 1 + c_1$, and we have

$$|F'_{k+1}(y)| \leq (c_0 k_0)^{k+1} \sum_{j=1}^{k+2} \binom{k+1}{j-1} |y|^{-1+j(1-m_0)-q}.$$

This shows the second part of the lemma. □

Now since

$$|D_x C_n| = \left| \frac{-nF_n + \sum_{j=0}^{n} \sum_{k=0}^{\infty} P_{n+k+1-j} F'_j x^{-k}}{x^{n+1}} \right| \lesssim \frac{|P_0(y)|}{|x|^{n+1} |y|^{m_0+q}}$$

(cf. 10), the estimate for x_{sing} follows immediately from integrating $D_x C_n$ from x_0 to $x_{sing} = C_n(\infty, x_{sing})$ along the simple singular solution path.

□

Remark 13. *The condition*

$$\frac{|P_k(y)|}{|P_0(y)|} \lesssim |y|^{-q}$$

is not the most general one for which there exists a formal constant of motion in a simple singular domain. However, this condition is frequently satisfied by ODEs that occur in applications (see §4). In such cases we can easily use (30) to find the position of the singularity (see e.g. (41)).

4. Example: the nonintegrable Abel equation (2)

To illustrate how to obtain information of the solution of a first order ODE using Theorem 4 and Proposition 11, we take as an example the nonintegrable Abel equation (2). Normalization is achieved by the transformation $x = -(9/5)A^2 t^{5/3}$, $A^3 = 1$, $u(t) = A^{3/5}(-135)^{1/5} x^{1/5} y(x)$ [17], yielding

$$y' = -3y^3 + \frac{1}{9} - \frac{y}{5x}. \tag{35}$$

Obviously (35) satisfies the assumptions in Theorem 4 and 11, with $P_0(y) = -3y^3 + \frac{1}{9}$ and $P_1(y) = -\frac{y}{5}$.

The three roots of P_0 are $\frac{1}{3}$, $\frac{(-1)^{2/3}}{3}$, and $\frac{(-1)^{4/3}}{3}$. It is known [17] that there exists a solution in the right half plane $\mathbb{H}$ that goes to the root $\frac{1}{3}$ as $x \to \infty$. Similarly, there are solutions that go to the other two roots in other regions, which we will explore in §4.3. In those cases, the behavior of the solution follows from Proposition 6 (see also [17]). However, there are also solutions that do not go to any of the three roots. In these cases, the formal constant of motion will be a useful tool to describe quantitatively the behavior of the solution.

4.1. **Constants of motion in R-domains.**

(cf. Definition 3). First we choose an elementary solution path along which the solution y to (35) turns around the root $\frac{1}{3}$ clockwise as shown in Fig 1 and 2.

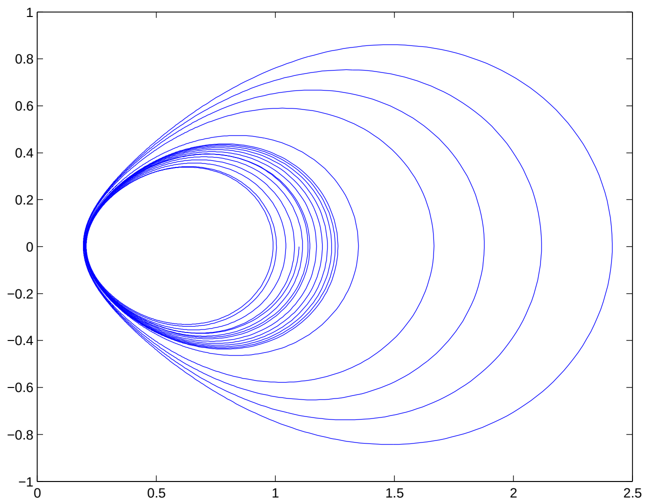

FIGURE 1. Solution $y(x)$ with $y_0 = 1.1$ along the line segments from 1+5i to 1.5+50i to 1.6+120i

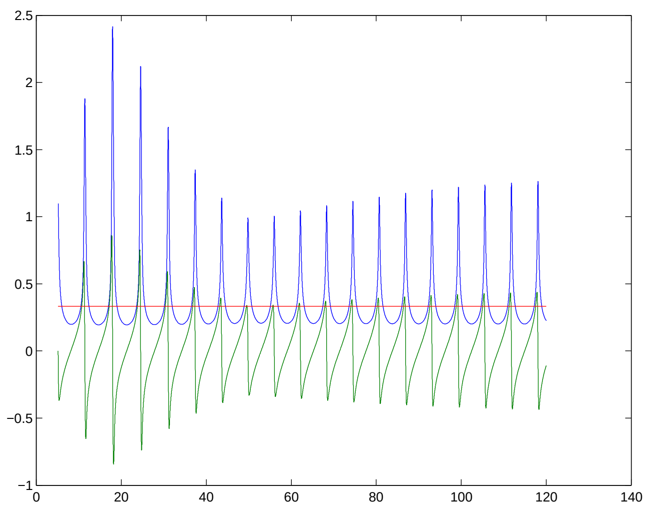

FIGURE 2. Real and imaginary parts of $y(x)$. The upper curve is the real part, the lower curve is the imaginary part, and the straight line is the root 1/3.

For simplicity we calculate the first two terms of the expansion (9). We have

(36)
$$F_0(y) = \int \frac{1}{-3y^3 + \frac{1}{9}} dy = \sqrt{3}\arctan\left(\frac{6y+1}{\sqrt{3}}\right) - \log(3y-1) + \frac{1}{2}\log(9y^2+3y+1)$$

(37)
$$a = \frac{\int_{\mathcal{C}} \frac{y}{5(-3y^3+\frac{1}{9})^2} dy}{\int_{\mathcal{C}} \frac{1}{-3y^3+\frac{1}{9}} dy} = \frac{1}{5}$$

(38)
$$F_1(y) = -\int \frac{\frac{1}{5} - \frac{y}{5(-3y^3+\frac{1}{9})}}{-3y^3+\frac{1}{9}} dy = \frac{1}{10}\left(\frac{54y^2}{1-27y^3} - 4\sqrt{3}\arctan\left(\frac{6y+1}{\sqrt{3}}\right)\right) + \frac{1}{25},$$

where the constant $\dfrac{1}{25}$ is found using (25).

We plot the first two orders of the formal constant of motion in Fig. 3.

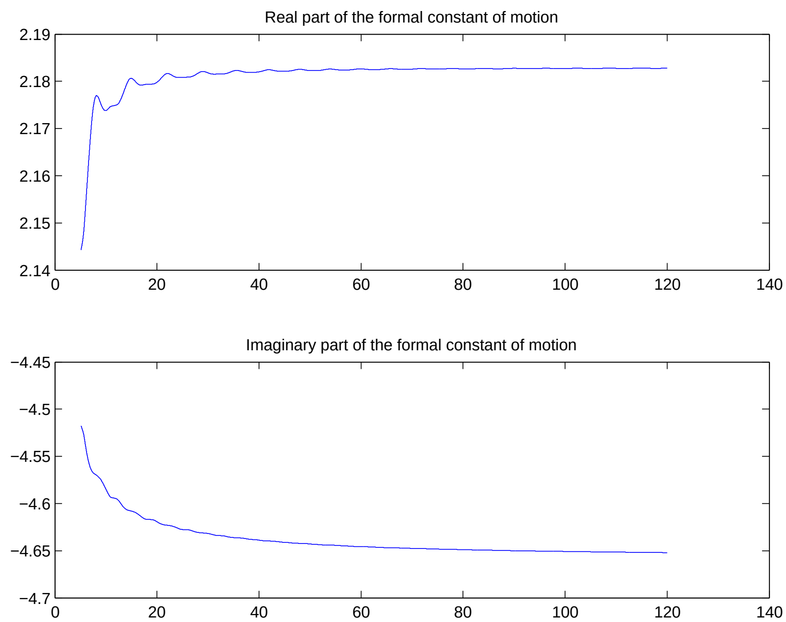

FIGURE 3. Formal constant of motion with F_0 and F_1.

Since this formal constant of motion is almost a constant along any path in the same R-domain, it can be used to find the solution asymptotically, writing

$$C = -x + \frac{1}{5}\log x + F_0(y) + \frac{F_1(y) + O(1/x)}{x}.$$

Placing the term $\log(3y-1)$ (cf. (36)) in the equation above on the left side and C on the right side, taking the exponential, and solving for y, we obtain

$$y = \frac{1}{3}\exp\Bigg(- C - x + \frac{1}{5}\log x + \left(\sqrt{3} - \frac{2\sqrt{3}}{5x}\right)\arctan\left(\frac{6y+1}{\sqrt{3}}\right) + \frac{1}{2}\log(9y^2+3y+1) + \frac{1}{x}\left(\frac{27y^2}{5(1-27y^3)} + \frac{1}{25} + O(1/x)\right)\Bigg) + \frac{1}{3}. \tag{39}$$

The reason for taking the exponential in (39) is to take care of the branching due to $\log x$, whereas the other log and arctan do not matter since the solution does not encircle their singularities. Equation (39) contains, in an implicit form, the solution y to two orders in x. y can be determined from this implicit equation in a number of ways; we chose, for simplicity to numerically solve the implicit equation using Newton's method. The solution is plotted in Fig. 4, where we take $C = 2.18 - 4.65i$ and calculate the solution for the second half of the path corresponding to $|x| > 61.4$. Note that the relative error is within 1.5%.

Since the accuracy of the formal constant of motion is unaffected by going along the solution path as long as $|x|$ is large, we can obtain quantitative behavior of the

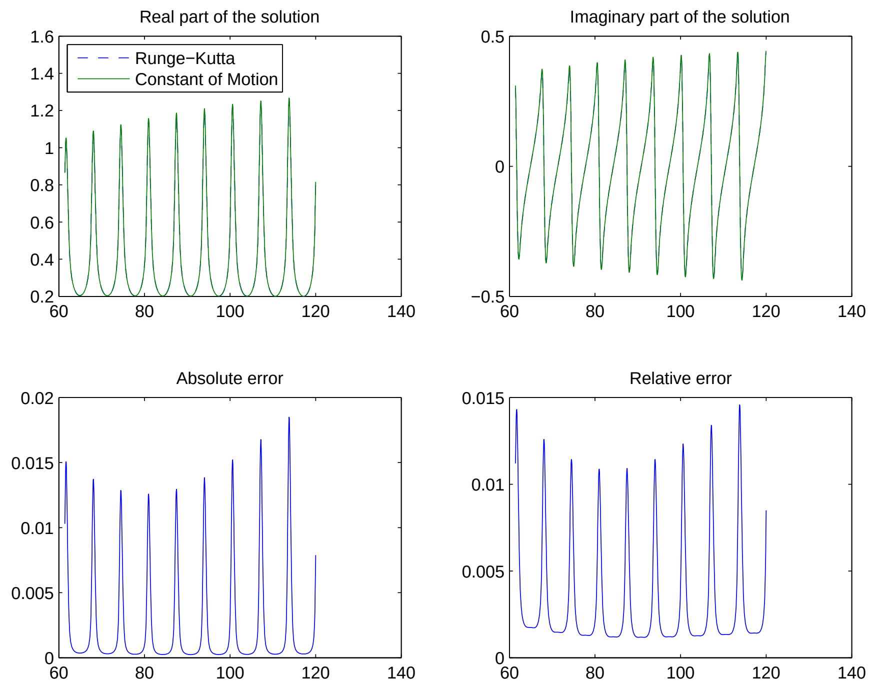

FIGURE 4. Comparison of solutions obtained numerically by the Runge-Kutta method and using the formal constant of motion .

solution for very large $|x|$. By contrast, in a numerical approach, the further one integrates along the path, the less accurate the calculated solution becomes.

4.2. **Finding the positions of the singularities.** We illustrate how to find singularities of the Abel's equation using Proposition 11. It is known [17] that there are only square root singularities, and they appear in two arrays.

For simplicity we choose a simple singular path along which y goes to $+\infty$.

According to Proposition 11 we have

$$F_0(y) = -\int_\infty^y \frac{1}{-3s^3+\frac{1}{9}}ds \tag{40}$$
$$= -\sqrt{3}\arctan(\frac{6y+1}{\sqrt{3}}) + \log(3y-1) - \frac{1}{2}\log(9y^2+3y+1) + \frac{\sqrt{3}\pi}{2}$$
$$F_1(y) = -\frac{1}{5}\int \frac{y}{(-3y^3+\frac{1}{9})^2}dy$$
$$= \frac{-\frac{54y^2}{1-27y^3} + 2\sqrt{3}\arctan(\frac{6y+1}{\sqrt{3}}) + 2\log(3y-1) - \log(9y^2+3y+1)}{10}.$$

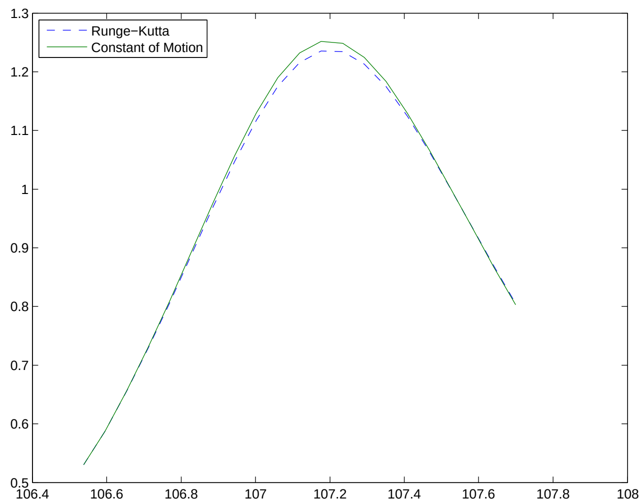

FIGURE 5. A small section of the left-top plot in Fig. 4.

Thus the position of the singularity is given by the formula

$$x_1 = C + o(1) = x_0 - \sqrt{3}\left(1 - \frac{1}{5x_0}\right)\left(\arctan\left(\frac{6y_0+1}{\sqrt{3}}\right) - \frac{\pi}{2}\right) \\ + \left(1 + \frac{1}{5x_0}\right)\left(\log(3y_0 - 1) - \frac{1}{2}\log(9y_0^2 + 3y_0 + 1)\right) - \frac{27y_0^2}{5x_0(1-27y_0^3)} + o(1), \tag{41}$$

where the initial condition (x_0, y_0) satisfies $|x_0|$ is large and y_0 is not close to any of the three roots. We note that the presence of the arctan in the leading order implies that the solutions remain quasi-periodic beyond the domain accessible to the methods in [17]. In (41) we have the freedom of choosing branch of log and arctan, which enables us to find arrays of singularities.

For example, the position of a singularity corresponding to the initial condition $x_0 = 10 + 60i$, $y_0 = 0.7 + 0.3i$, calculated using (41) is $x_1 = 9.80628 + 60.2167i$, which is accurate with six significant digits, as checked numerically.

The detailed behavior of the solution near the singularity can be found by expanding the right hand side of (41). We omit the calculation here since there are many other methods to determine this behavior (cf. [17]) and it is of lesser importance to the paper.

4.3. **Connecting regions of transseries.** We choose a path consisting of line segments The path in x consists of line segments connecting $50i$, 50, $-50i$, -50, $50i$, 50, $-50i$, and $-50(\sqrt{3}+i)$. This corresponds to an angle of 2π in the original variable, with initial condition $y(50i) = 0.6$.

Along this path, the solution of (35) approaches all three complex cube roots of $1/27$. For instance, the root $1/3$ is approached when x traverses the first quadrant along the first segment, the root $(-1)^{4/3}/3$ is approached when x goes to the lower half plane, and the root $(-1)^{2/3}/3$ is approached when x goes back to the upper half plane. Some of these values are approached more than once along the entire

path. This behavior can easily be shown using the phase portrait of G_0, cf. (16) and Corollary 8.

Note that along a straight line $x = x_0 + xe^{ti}$ where the angle t is fixed the leading term (with only G_0 on the right hand side) of the ODE (35) can be written as

$$\frac{dy}{dx} = e^{ti}\left(-3y^3 - \frac{y}{5(x_0 + xe^{ti})} + \frac{1}{9}\right).$$

Denoting $y_1 = \operatorname{Re} y$ and $y_2 = \operatorname{Im} y$, we have

$$\begin{cases} \dfrac{dy_1}{dx} = -3y_1^3 \cos t - 3y_2^3 \sin t + 9y_1 y_2^2 \cos t + 9y_1^2 y_2 \sin t + \dfrac{\cos t}{9} \\ \dfrac{dy_2}{dx} = -3y_1^3 \sin t + 3y_2^3 \cos t + 9y_1 y_2^2 \sin t - 9y_1^2 y_2 \cos t + \dfrac{\sin t}{9} \end{cases}$$

We can then analyze the phase portraits. For the purpose of illustration, we show some of them in Fig 6 and 7.

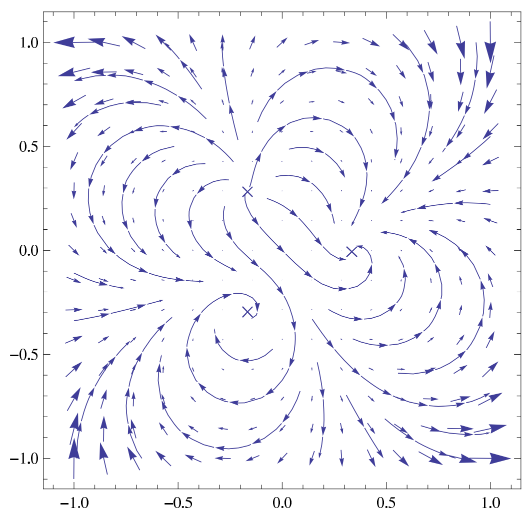

FIGURE 6. Phase portrait of Re (y) and Im (y) for $t = -\pi/4$. The "$\times$" marks are the three roots.

On the line segment connecting $50i$ and 50, it is clear that the initial condition 0.6 is in the basin of attraction of 1/3 (cf. Fig. 6).

Since the only stable equilibrium is $a_0 = \dfrac{(-1)^{4/3}}{3}$, on the line segment connecting 50 and $-50i$ the solution converges to a_0 (cf. Fig. 7).

Numerical calculations confirm this, (cf. Fig 8).

Note 14 (Absence of limit cycles)**.** *Finally, note that there cannot be a limit cycle in the phase portraits drawn if x goes along a straight line. If the solution y approaches a limit cycle, it must lie in an R-domain. Thus the formal constant motion formula (9) is valid, and the first term F_0 specifies a direction for x. If x goes strictly along this direction towards ∞ then the term $a \log x$, which does not vanish in our case, will go to ∞, contradicting the results about the constant of motion. On the other hand, if x goes in a different direction, then $-x + F_0(y)$ goes to ∞ much faster than $a \log x$, again a contradiction.*

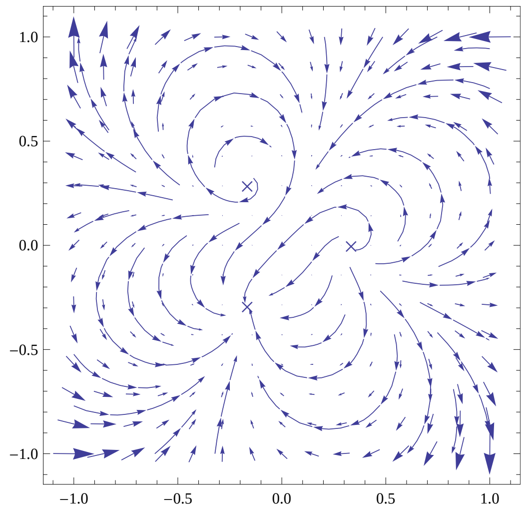

FIGURE 7. Phase portrait of $\mathrm{Re}\,(y)$ and $\mathrm{Im}\,(y)$ for $t = 5\pi/4$.

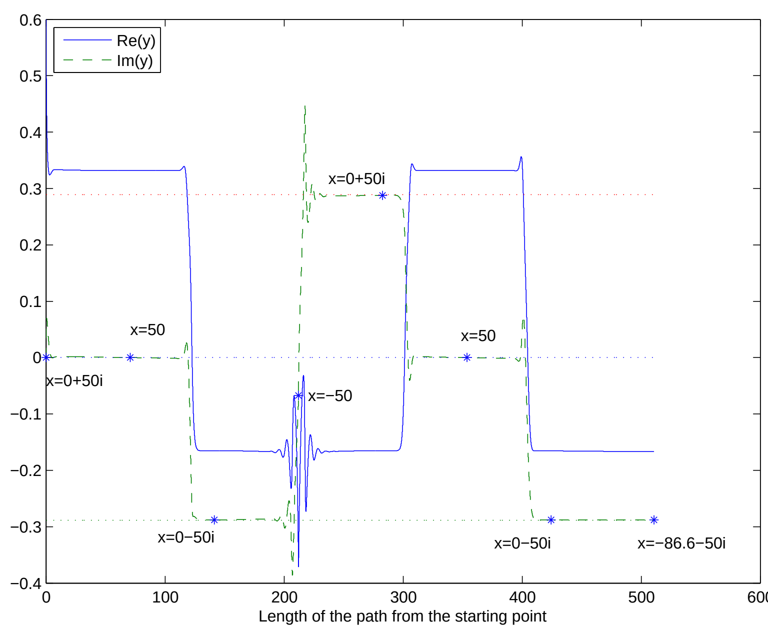

FIGURE 8. Behavior of the solution across transseries regions. Dotted horizontal lines are the imaginary parts of the three roots. The horizontal line is arclength. The path in x consists of line segments connecting $50i$, 50, $-50i$, -50, $50i$, 50, $-50i$, and $-50(\sqrt{3}+i)$. This corresponds to an angle of 2π in the original variable.

4.4. **Extension to higher orders.** For higher orders, such as the Painlevé equations P1 and P2, a similar procedure works, though the details are quite a bit more

complicated, and we leave them for a subsequent work. We illustrate, without proofs, the results for $P1$, $y'' = 6y^2 + z$. Now, there are two asympotic constants of motion, as expected. The normal form we work with is $u'' + u'x^{-1} - u - u^2/2 - 392x^{-4}/625 = 0$. Denoting by s the "energy of elliptic functions" $s = u'^2/2 - u^3/3 + u^2$ (it turns out that s is one of the bicharacteristic variables of the sequence of now PDEs governing the terms of the expansion; thus the pair (u, s) is preferable to (u, u')), one constant of motion has the asymptotic form

$$C_1 = x - L(s, u) + x^{-1}K_1(s, u) + \cdots$$

In the above, denoting $R = \sqrt{u^3/3 + u^2 + s}$, L is an incomplete elliptic integral, $L = \int R^{-1}(s, u)du$ and the integration is following a path winding around the zeros of R. The functions K_1, K_2, $\cdots$ have similar but longer expressions. We note the absence of a term of the form $a \log x$ (the reason for this is easy to see once the calculation is performed). A second constant can now be obtained by reduction of order and applying the first order techniques, or better, by the "action-angle" approach described in the introduction. It is of the form

$$C_2 = xJ(s) + [L(s)J(u, s) - J(s)L(u, s)] + x^{-1}\tilde{K}_1 + \cdots$$

where $J(u, s) = \int R(s, u)du$; when the variable u is missing from $J(u, s)$ or $R(u, s)$, this simply means that we are dealing with complete elliptic integrals. There is directionality in the asymptotics, as the loops encircling the singularities need to be rigidly chosen according to the asymptotic direction studied. A slightly different representation allows us to calculate the constants to all orders. Because of directionality, a different asymptotic formula exists and is more useful for the "lateral connection", that is, for calculating the solution along a circle of fixed but large radius, which will be detailed in a separate paper, as part of the Painlevé project, see e.g. [9].

5. Acknowledgments

The authors are very grateful to R. Costin for a careful reading of the manuscript and numerous useful suggestions. OC's work was partially supported by the NSF grants DMS 0807266 and DMS 0600369.

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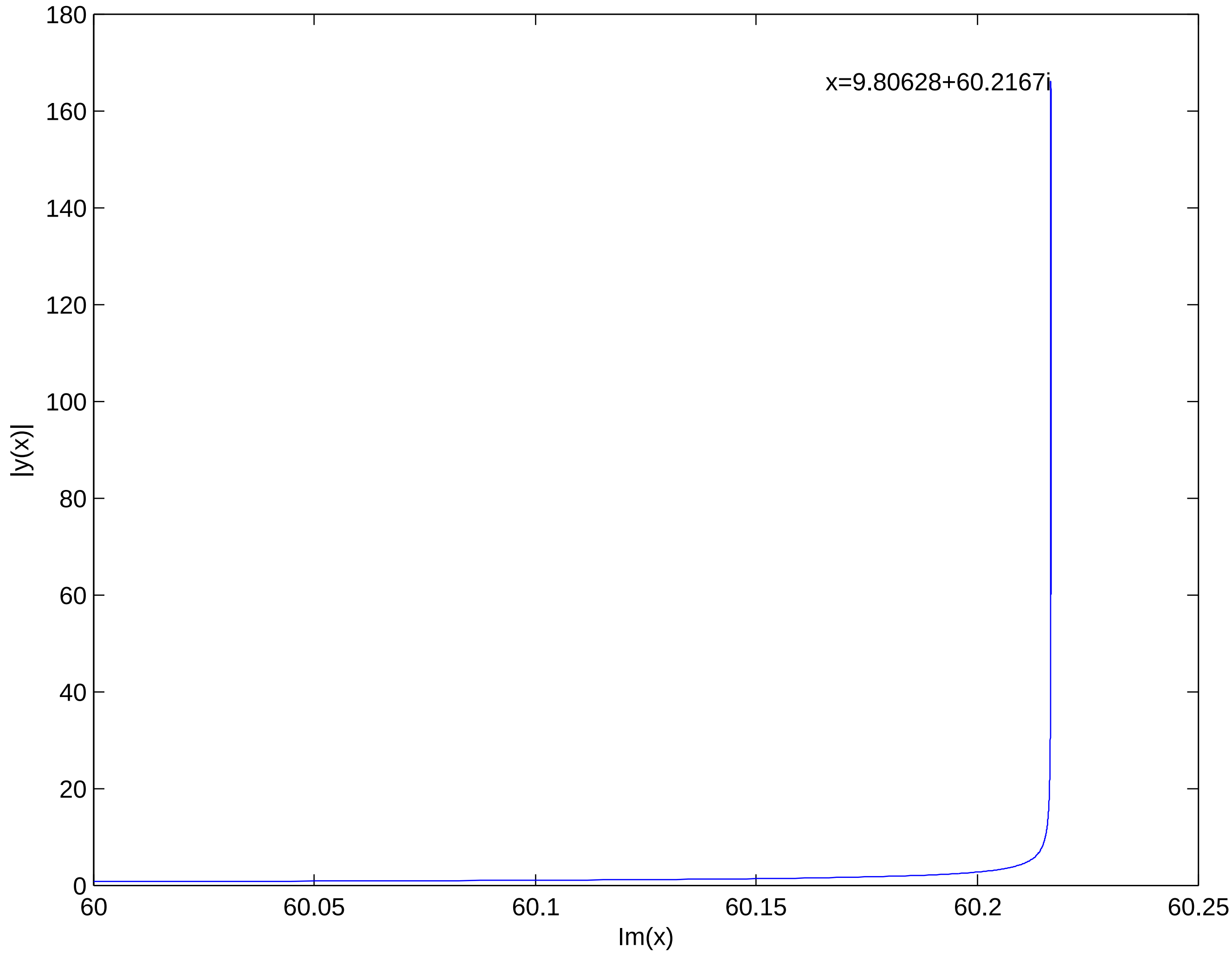

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60.1
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